Selectors for dense subsets of function spaces

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Abstract

Let $\mathrm{USC}^{\star}_p(X)$ be the topological space of real upper semicontinuous bounded functions defined on X with the subspace topology of the product topology on ${}^X\mathbb{R}$. $\tilde{\Phi}^{\uparrow}, \tilde{\Psi}^{\uparrow}$ are the sets of all upper sequentially dense, upper dense or pointwise dense subsets of $\mathrm{USC}^{\star}_p(X)$, respectively. We prove several equivalent assertions to that that $\mathrm{USC}^{\star}_p(X)$ satisfies the selection principles $\mathrm{S}_1(\tilde{\Phi}^{\uparrow}, \tilde{\Psi}^{\uparrow})$, including a condition on the topological space X.
We prove similar results for the topological space $\mathrm{C}^{\star}_p(X)$ of continuous bounded functions.
Similar results hold true for the selection principles $\mathrm{S}_{fin}(\tilde{\Phi}^{\uparrow}, \tilde{\Psi}^{\uparrow})$.

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1. Introduction

We shall study the relationship between selection properties of covers of a topological space X and selection properties of dense subsets of the set $\mathrm{USC}^{\star}_p(X)$ of all bounded upper semicontinuous functions on X and the set of all bounded continuous functions $C^*_p(X)$ on X with the topology of pointwise convergence. Similar problems were studied by M. Scheepers [21, 22], J. Haleš [7], A. Bella, M. Bonanzinga, M. Matveev [1], M. Sakai [16] – [18], W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki [9], and authors of this paper [3, 4, 12, 13, 14, 15] for the set $C_p(X)$ of all real-valued continuous functions with the topology of pointwise convergence.

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We shall find equivalent conditions in the term of some properties of density of subsets of $\mathrm{USC}_p^\star(X)$ and the covering properties $\mathrm{S}_1(\Phi, \Psi)$ for $\Phi, \Psi = \mathcal{O}, \Omega, \Gamma$. The main results are presented in Theorems 5.2, 5.4, 6.1, 6.4 and 7.1. Also we present similar Theorems 8.2, 8.3 and 8.4 for bounded continuous functions.

2. Terminology and notations

In the paper (X, τ) is an infinite Hausdorff topological space, τ is the topology, i.e., the set of open sets.

We shall follow the standard terminology and notation as that of [5]. We recall some notions.

If $A \subseteq X$ then the **sequential closure** $[A]_{seq}$ **of** A is the set of all limits of sequences from A.

From some technical reasons, a cover is called o**-cover**. A cover $\mathcal{U}$ of X is an ω**-cover** if $X \notin \mathcal{U}$ and for any finite set $F \subseteq X$ there exists $U \in \mathcal{U}$ that contains F as a subset. An infinite cover $\mathcal{U}$ is a γ**-cover** if for every $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite. By $\mathcal{O}(X)$, $\Omega(X)$, $\Gamma(X)$, or simply $\mathcal{O}$, Ω, Γ when the topological space X is understood, we denote the family of all open, open ω and open γ-covers, respectively.

Let us recall the a cover $\mathcal{V}$ is a **refinement of a cover** $\mathcal{U}$ iff

$$(\forall V \in \mathcal{V})(\exists U \in \mathcal{U}\, V \subseteq U. \tag{1}$$

If φ denotes one of the symbols o, ω or γ, then a φ-cover $\mathcal{U}$ is **shrinkable**, if there exists an open φ-cover $\mathcal{W}$ such that

$$(\forall W \in \mathcal{W})(\exists U_W \in \mathcal{U})\, \overline{W} \subseteq U_W. \tag{2}$$

The family $\{U_W : W \in \mathcal{W}\} \subseteq \mathcal{U}$ is a φ-cover as well. The family of all open shrinkable o-, ω- or γ-covers of X will be denoted by $\mathcal{O}^{sh}(X)$, $\Omega^{sh}(X)$ and $\Gamma^{sh}(X)$, or simply $\mathcal{O}^{sh}$, Ω^{sh} and Γ^{sh}, respectively.

The set ${}^X\mathbb{R}$ of all real function defined on X is endowed with the product topology. Thus, a typical neighborhood of a function $g \in {}^X\mathbb{R}$ is the set

$$V = \{h \in {}^X\mathbb{R} : |h(x_j) - g(x_j)| < \varepsilon : j = 0, \ldots, k\} \tag{3}$$

where ε is a positive real and $x_0, \ldots, x_k \in X$. A sequence of real functions $\langle f_n : n \in \omega\rangle$ converges to a real function f in this topology if it converges pointwise, i.e., if $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$.

Similarly as in [3] we introduce the following properties of a family F of real functions and a function $h \in {}^X\mathbb{R}$:

- $(\mathcal{O}_h)$ $h(x) \in \overline{\{f(x) : f \in F\}}$ for every $x \in X$.
- (Ω_h) $h \notin F$ and $h \in \overline{F}$ *in the topology of* ${}^X\mathbb{R}$.
- (Γ_h) F is infinite and for every $\varepsilon > 0$ and for every $x \in X$ the set $\{f \in F : |f(x) - h(x)| \geq \varepsilon\}$ is finite.

Let $H \subseteq {}^X\mathbb{R}$. We set

$$\Phi_h(H) = \{F \subseteq H : F \text{ possesses } (\Phi_h) \wedge (\forall f \in F)\,(f \geq h \wedge f - h \in H)\}.$$

For a real a, we denote by $\mathbf{a}$ the constant function on X with value a. For simplicity for $g \in {}^X\mathbb{R}$, instead of $g + \mathbf{a}$ or $g - \mathbf{a}$ we shall write $g + a$ or $g - a$, respectively. Similarly for $\min\{\mathbf{a}, g\}$ or $\max\{\mathbf{a}, g\}$. If $F \subseteq {}^X\mathbb{R}$, then

$$F^+ = \{f \in F : f \geq 0\}, \quad F^* = \{f \in F : f \text{ is bounded}\}.$$

$\mathrm{C}_p(X)$ or $\mathrm{USC}_p(X)$ denote the set of all real continuous or upper semicontinuous functions[1] defined on the topological space X. Instead of $\mathrm{C}_p(X)^*$ or $\mathrm{USC}_p(X)^*$ we write $\mathrm{C}^\star_p(X)$ or $\mathrm{USC}^\star_p(X)$, respectively.

A set $F \subseteq H \subseteq {}^X\mathbb{R}$ is **sequentially dense in** H if $H \subseteq [F]_{seq}$. The set $F \subseteq H \subseteq {}^X\mathbb{R}$ is **countably dense in** H if for every function $f \in H$ there exists a countable set $G \subseteq F$ such that $f \in \overline{G}$. As obviously, the set F is **dense in** H if $H \subseteq \overline{F}$. Finally, the set F is **pointwise dense in** $A \subseteq \mathbb{R}$ if $A \subseteq \overline{\{f(x) : f \in F\}}$ for each $x \in X$ (**1-dense set** in terminology of [14, 15]). We set

$$\begin{aligned}
\mathcal{S}(H) &= \{F \subseteq H : F \text{ is sequentially dense in } H\},\\
\mathcal{D}(H) &= \{F \subseteq H : F \text{ is dense in } H\},\\
\mathcal{P}(H) &= \{F \subseteq H : F \text{ is pointwiae dense in } H\}.
\end{aligned}$$

Then

$$\mathcal{S}(H) \subseteq \mathcal{D}(H) \subseteq \mathcal{P}(H).$$

Evidently a sequentially dense set is countably dense as well. By Tong Theorem, see, e.g., [5], if X is perfectly normal topological space then every (bounded) upper semicontinuous function is a limit of a non-increasing sequence of (bounded) continuous functions. Thus for a perfectly normal topological space X the set $\mathrm{C}_p(X)$ is sequentially dense in $\mathrm{USC}_p(X)$. Then the set $\mathrm{C}^\star_p(X)$ is sequentially dense in $\mathrm{USC}^\star_p(X)$ as well.

The authors suppose that the answer to the following problem is negative.

Problem 2.1. *Let X be a normal topological space. If $\mathrm{C}^\star_p(X)$ is sequentially dense in $\mathrm{USC}^\star_p(X)$ then X is perfectly normal?*

A set $F \subseteq H \subseteq {}^X\mathbb{R}$ is **upper sequentially dense in** H if for every $f \in H$ there exists a sequence $\langle h_n : n \in \omega \rangle$ of elements of F such that $h_n \to f$, $h_n \geq f$ and $h_n - f \in H$ for each $n \in \omega$. A set $F \subseteq H$ is **upper dense in** H if for every $f \in H$ the set $\{h \in F : h \geq f \wedge h - f \in H\}$ is dense in the set $\{h \in H : h \geq f\}$. Evidently, every upper sequentially dense in H is also upper dense in H.

One can easily see that if a set $F \subseteq \mathrm{USC}^\star_p(X)$ is upper dense in $\mathrm{USC}^\star_p(X)$, then for every continuous function f the set of upper semicontinuous functions $\{h - f : h \in F \wedge h \geq f\}$ is upper dense in $\mathrm{USC}^\star_p(X)^+$. If the set F is upper

[1] A function $f : X \longrightarrow \mathbb{R}$ is said to be upper semicontinuous if for every real a the set $\{x \in X : f(x) < a\}$ is open.

sequentially dense in $\mathrm{USC}_p^\star(X)$, then for every $f \in \mathrm{C}_p^\star(X)$ the set $\{h - f : h \in F \wedge h \geq f\}$ is upper sequentially dense in $\mathrm{USC}_p^\star(X)^+$. We set

$$\begin{aligned}\mathcal{S}^\uparrow(H) &= \{F \subseteq H : F \text{ is upper sequentially dense in } H\},\\ \mathcal{D}^\uparrow(H) &= \{F \subseteq H : F \text{ is upper dense in } H\},\\ \mathcal{P}^\uparrow(H) &= \mathcal{P}(H)\end{aligned}$$

Then

$$\mathcal{S}^\uparrow(H) \subseteq \mathcal{D}^\uparrow(H) \subseteq \mathcal{P}^\uparrow(H).$$

We introduce the following notations. If $\Phi = \Gamma$ then $\tilde{\Phi} = \mathcal{S}$. If $\Phi = \Omega$ then $\tilde{\Phi} = \mathcal{D}$ and if $\Phi = \mathcal{O}$ then $\tilde{\Phi} = \mathcal{P}$. Similarly for $\tilde{\Phi}^\uparrow$.

Note that by definitions we have immediately

$$(\forall F \in \tilde{\Phi}(H)^\uparrow)(\forall h \in H)(\exists G \subseteq F)\, G \in \Phi_h(H). \tag{4}$$

3. Covering Properties and Selection Properties

We recall some notions introduced in [21] and [9]. Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(Y)$ for some set Y.

The **covering property** $\mathrm{S}_1(\mathcal{A}, \mathcal{B})$ means the following: for any sequence of sets $\langle \mathcal{U}_n \in \mathcal{A} : n \in \omega\rangle$ for every n there exists an $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \omega\} \in \mathcal{B}$.

The **covering property** $\mathrm{S}_{fin}(\mathcal{A}, \mathcal{B})$ means the following: for any sequence of sets $\langle \mathcal{U}_n \in \mathcal{A} : n \in \omega\rangle$, for every n there exists a finite set $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{n\in\omega} \mathcal{V}_n \in \mathcal{B}$.

For $\Phi, \Psi = \mathcal{O}, \Omega, \Gamma$, if the covering property $\mathrm{S}_1(\Phi(X), \Psi(X))$ holds true, we say that X is an $\mathrm{S}_1(\Phi, \Psi)$-space. Similarly for S_{fin}.

If $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}({}^X\mathbb{R})$ for some X are sets of sets of functions, we prefer to speak about the **selection principle** $\mathrm{S}_1(\mathcal{F}, \mathcal{G})$ or $\mathrm{S}_{fin}(\mathcal{F}, \mathcal{G})$. For $\Phi, \Psi = \mathcal{O}, \Omega, \Gamma$ and a family $H \subseteq {}^X\mathbb{R}$ we say that H satisfies the selection principle $\mathrm{S}_1(\Phi_h, \Psi_h)$ if $\mathrm{S}_1(\Phi_h(H), \Psi_h(H))$ holds true.

In [3] the author (Theorems 6.2 and 6.5 part 5))[2] has proved

Theorem 3.1 (L. Bukovský). *Assume that Φ is one of the symbols Ω and Γ, and Ψ is one of the symbols $\mathcal{O}$, Ω, Γ. Then for any couple $\langle \Phi, \Psi\rangle$ different from $\langle \Omega, \mathcal{O}\rangle$, a topological space X is an $\mathrm{S}_1(\Phi, \Psi)$-space if and only if $\mathrm{USC}_p^\star(X)$ satisfies the selection principle $S_1(\Phi_{\mathbf{0}}, \Psi_{\mathbf{0}})$.*

If a topological space X possesses the property[3] (ε) then the equivalence holds true also for the couple $\langle \Omega, \mathcal{O}\rangle$.

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[2]The Theorem was formulated and proved for $\mathrm{USC}_p(X)^+$. One can easily see that it holds equally for $\mathrm{USC}_p^\star(X)^+$.

[3]The property (ε) was introduced and investigated in [6]: any ω-cover contains a countable ω-subcover.

Corollary 3.2 (A.V. Osipov). *Assume that Φ is one of the symbols Ω and Γ, and Ψ is one of the symbols $\mathcal{O}$, Ω, Γ. Then for any couple $\langle\Phi,\Psi\rangle$ different from $\langle\Omega,\mathcal{O}\rangle$, a topological space X is an $\mathrm{S}_1(\Phi,\Psi)$-space if and only if for every $h\in\mathrm{USC}^\star_p(X)$ the family $\mathrm{USC}^\star_p(X)$ satisfies the selection principle $S_1(\Phi_h,\Psi_h)$.*

If a topological space X possesses the property (ε) then the equivalence holds true also for the couple $\langle\Omega,\mathcal{O}\rangle$.

Proof:

The implication from right to left follows by Theorem 3.1.

Let $h\in\mathrm{USC}^\star_p(X)$. If $F_n\in\Phi_h(\mathrm{USC}^\star_p(X))$ then

$$F_n-h=\{g\in\mathrm{USC}^\star_p(X)^+ : (\exists f\in F_n)\, g=f-h\}$$

belongs to $\Phi_{\mathbf{0}}(\mathrm{USC}^\star_p(X))$. Thus we can apply the selection principle $\mathrm{S}_1(\Phi_{\mathbf{0}},\Psi_{\mathbf{0}})$. For every $n\in\omega$ we obtain $g_n\in F_n-h$ such that $\{g_n : n\in\omega\}\in\Psi_{\mathbf{0}}$. Then $\{g_n+h : n\in\omega\}\in\Psi_h$. $\square$

By Theorem 4.1, part 2) of [4] similar results hold true also for the principle S_{fin}.

Theorem 3.3 (L. Bukovský). *Assume that Φ is one of the symbols Ω and Γ, and Ψ is one of the symbols $\mathcal{O}$, Ω, Γ. Then for any couple $\langle\Phi,\Psi\rangle$ different from $\langle\Omega,\mathcal{O}\rangle$, a topological space X is an $\mathrm{S}_{fin}(\Phi,\Psi)$-space if and only if $\mathrm{USC}^\star_p(X)$ satisfies the selection principle $\mathrm{S}_{fin}(\Phi_{\mathbf{0}},\Psi_{\mathbf{0}})$.*

If a topological space X possesses the property (ε) then the equivalence holds true also for the couple $\langle\Omega,\mathcal{O}\rangle$.

Corollary 3.4. *Assume that Φ is one of the symbols Ω and Γ, and Ψ is one of the symbols $\mathcal{O}$, Ω, Γ. Then for any couple $\langle\Phi,\Psi\rangle$ different from $\langle\Omega,\mathcal{O}\rangle$, a topological space X is an $\mathrm{S}_{fin}(\Phi,\Psi)$-space if and only if for every $h\in\mathrm{USC}^\star_p(X)$ the family $\mathrm{USC}^\star_p(X)$ satisfies the selection principle $S_{fin}(\Phi_h,\Psi_h)$.*

If a topological space X possesses the property (ε) then the equivalence holds true also for the couple $\langle\Omega,\mathcal{O}\rangle$.

We shall need also Theorem 6.3 of [3] which reads as follows:

Theorem 3.5 (L. Bukovský). *Assume that Φ is one of the symbols Ω, Γ and Ψ is one of the symbols $\mathcal{O}$, Ω, Γ. Then for any couple $\langle\Phi,\Psi\rangle$ different from $\langle\Omega,\mathcal{O}\rangle$ a normal topological space X is an $\mathrm{S}_1(\Phi^{sh},\Psi)$-space if and only if $\mathrm{C}^\star_p(X)$ satisfies the selection principle $S_1(\Phi_{\mathbf{0}},\Psi_{\mathbf{0}})$.*

If a topological space X possesses the property (ε) then the equivalence holds true also for the couple $\langle\Omega,\mathcal{O}\rangle$.

Note the following:

$$\begin{aligned} &if\ \mathrm{C}^\star_p(X)\ satisfies\ the\ selection\ principle\ \mathrm{S}_1(\Phi_{\mathbf{0}},\Psi_{\mathbf{0}}),\\ &\quad then\ it\ satisfies\ \mathrm{S}_1(\Phi_h,\Psi_h)\ for\ each\ h\in\mathrm{C}^\star_p(X). \end{aligned} \tag{5}$$

4. Separability of spaces of real functions

Since the selection principle usually produces a countable accordingly dense subset of the considered space of functions we present some conditions on the topological space for corresponding separability of the functions spaces.

We recall that the i-**weight** $iw(X)$ of a topological space X is the smallest infinite cardinal number κ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than κ.

By the classical result of P. Urysohn [23] one can easily see that a Tychonoff space $\langle X, \tau\rangle$ has $iw(X) = \aleph_0$ if and only if there exists a metric ρ on X such that $\langle X, \rho\rangle$ is separable and the topology τ_ρ induced by the metric ρ is weaker than the the topology τ, i.e. $\tau_\rho \subseteq \tau$.

The classical result is

Theorem 4.1 (N. Noble [11]). *Let X be a Tychonoff space. The topological space $\mathrm{C}_p^\star(X)$ is separable if and only if $iw(X) = \aleph_0$.*

Recall that $B_1(X)$ is a set of all first Baire class functions i.e., pointwise limits of continuous functions, defined on a space X. It is well-known that $\mathrm{USC}_p^\star(X) \subset B_1(X)$ for a perfectly normal space X.

A topological space $\langle X, \tau\rangle$ has the V**-property**, if there exist a metric ρ on X such that $\langle X, \rho\rangle$ is a separable metric space, the topology τ_ρ induced by the metric ρ is weaker than the topology τ and any cozero set in the topology τ is an F_σ set in the topology τ_ρ.

In [24] N.V. Velichko proved the following result.

Theorem 4.2 (N.V. Velichko). *Let X be a Tychonoff topological space. Then the following are equivalent*

a) *The topological space $\mathrm{C}_p^\star(X)$ is sequentially separable.*

b) *X has the V-property.*

c) *There exists a countable set S of bounded continuous functions such that $B_1^*(X) = [S]_{seq}$.*

Recall that a set is locally closed if it is an intersection of an open and a closed set. A topological space $\langle X, \tau\rangle$ has the OB**-property**, if there exists a metric ρ on X such that $\langle X, \rho\rangle$ is separable, the topology τ_ρ induced by the metric ρ is weaker than τ and any locally closed subset of $\langle X, \tau\rangle$ is an F_σ-set in the topology τ_ρ.

Note that OB-property implies V-property.

Theorem 4.3. *A Tychonoff topological space $\langle X, \tau\rangle$ possesses the OB-property if and only if there exists a countable set $S \subseteq \mathrm{C}_p^\star(X, \tau)$ such that S is sequentially dense in $\mathrm{USC}_p^\star(X, \tau)$.*

Proof:
($\rightarrow$). Assume that X has the OB-property, i.e. there a metric ρ on X with corresponding properties. Let $f \in \mathrm{USC}_p^\star(X,\tau)$, $a<b$ being reals. Since

$$(a,b) = \bigcup_{j\in\omega} \big((-\infty,b)\cap[a+2^{-j},+\infty)\big)$$

and

$$f^{-1}((-\infty,b)\cap[a+2^{-j},+\infty)) = f^{-1}((-\infty,b))\cap f^{-1}([a+2^{-j},+\infty))$$

is a locally closed subset of X in the topology τ, it follows that $f^{-1}(W)$ is an F_σ-set in the topology τ_ρ for any open W of $\mathbb{R}$. Thus by the Lebesgue – Hausdorff Theorem, see e.g. [10], we obtain that $f \in B_1(X,\rho)$. Thus

$$\mathrm{USC}_p^\star(X,\tau) \subseteq B_1(X,\rho).$$

By Theorem 4.2, there exists a countable set $S \subseteq \mathrm{C}_p^\star(X,\rho)$ such that $[S]_{seq} = B_1^*(X,\rho)$. Since

$$\mathrm{C}_p^\star(X,\rho) \subset \mathrm{C}_p^\star(X,\tau) \subset \mathrm{USC}_p^\star(X,\tau) \subset B_1(X,\rho),$$

it follows that $S \subset \mathrm{C}_p^\star(X,\tau)$ is a countable sequentially dense subset of $\mathrm{USC}_p^\star(X,\tau)$.
($\leftarrow$) Assume that $S=\{f_i : i\in\omega\} \subset \mathrm{C}_p^\star(X,\tau)$ is a sequentially dense subset of $\mathrm{USC}_p^\star(X,\tau)$.

Let τ' be the topology on X with basis

$$\left\{\bigcap_{i\in A} f_i^{-1}((a_i,b_i)) : A \in [\omega]^{<\omega} \wedge a_i,b_i \in \mathbb{Q} \wedge a_i<b_i \text{ for } i\in A\right\}.$$

The topology τ' has a countable basis, is regular, therefore by the Urysohn Theorem [23] there exists a metric ρ on X such that $\tau'=\tau_\rho$. Moreover, $f_i \in \mathrm{C}_p^\star(X,\rho)$ for every $i\in\omega$.

We claim that any locally closed set D of $\langle X,\tau\rangle$ is an F_σ-set of $\langle X,\rho\rangle$. Let $D=U\cap F$ where U is an open set of $\langle X,\tau\rangle$ and F is a closed set of $\langle X,\tau\rangle$. Define the function h as follows: $h(x)=0$ for $x\in U$ and $h(x)=2$ for $x\in X\setminus U$. By the definition of h, $U=h^{-1}(\{0\})$. Note that $h\in USC_p^*(X,\tau)$. Hence there exists an increasing sequence of integers $\langle n_k : k\in\omega\rangle$ such that $\langle f_{n_k} : k\in\omega\rangle$ converges to h. It follows that

$$U = h^{-1}(\{0\}) = \bigcup_{j\in\omega}\bigcap_{k>j} f_{n_k}^{-1}([-1,1])$$

and hence U is an F_σ-set of $\langle X,\rho\rangle$.

Similarly, we define the function g as follows: $g(x)=2$ for $x\in F$ and $g(x)=0$ for $x\in X\setminus F$. By the definition of g, $F=g^{-1}(\{2\})$. Note that $g\in$

$USC_p^*(X,\tau)$ and hence there is an increasing sequence of integers $\langle m_k : k \in \omega\rangle$ such that $\langle f_{m_k} : k \in \omega\rangle$ converges to g. Equally as above we obtain

$$F = g^{-1}(\{2\}) = \bigcup_{j\in\omega}\bigcap_{k>j} f_{m_k}^{-1}([1,+\infty))$$

and hence F is an F_σ-set of $\langle X,\rho\rangle$ as well.

It follows that $D = U \cap F$ is an F_σ-set of $\langle X,\rho\rangle$. $\square$

Corollary 4.4. *Let X be a Tychonoff topological space with the OB-property. Then $USC_p^*(X)$ is sequentially separable.*

Problem 4.5. *Is there a topological space X such that $USC_p^*(X)$ is sequentially separable, but X does not have the OB-property?*

Recall that a space is *perfect* if every open subset is an F_σ-subset [8]. Note that a topological space with the OB-property is perfect. However, the OB-property is stronger than to be perfect.

Theorem 4.6. *There is a perfect space which has not the OB-property.*

Proof: Denote by $\mathbb{S}$ the Sorgenfrey line. Then the space $\mathbb{S}^2$ is perfect (by Lemma 2.3 in [8]). We claim that $\mathbb{S}^2$ has not the OB-property. Consider the set $D = \{(x,-x) : x \in \mathbb{S}\}$. Then any subset B of D is a locally closed subset of $\mathbb{S}^2$. Suppose that f is a condensation from the space $\mathbb{S}^2$ on a separable metric space Y, such that $f(B)$ is an F_σ-set of Y for any locally closed set B of $\mathbb{S}^2$. Then $|f(D)| = \mathfrak{c}$ and $f(D)$ is a separable metrizable subspace of $f(\mathbb{S}^2)$ such that for any subset $Q \subset f(D)$ the set Q is an F_σ-set of $f(D)$, which is impossible (see, e.g., [10]). $\square$

5. Dense selectors of $\mathbf{USC_p^*(X)}$

We shall use the following families of functions. Let $\mathcal{U}$ be a cover. We set

$$S(\mathcal{U}) = \{f_{U,g} + g : U \in \mathcal{U} \wedge g \in \mathrm{USC}_p^\star(X)\}, \tag{6}$$

where

$$f_{U,g}(x) = \begin{cases} 0 & \text{if } x \in U, \\ 1 + \sup|g| & \text{otherwise.} \end{cases} \tag{7}$$

We show the basic properties of the families $S(\mathcal{U})$.

Lemma 5.1.

1) *If $\mathcal{U}$ is an open ω-cover, then the family $S(\mathcal{U}) \subseteq \mathrm{USC}_p^\star(X)$ is upper dense in $\mathrm{USC}_p^\star(X)$.*

2) *If $\mathcal{U}$ is an open γ-cover, then the family $S(\mathcal{U}) \subseteq \mathrm{USC}_p^\star(X)$ is upper sequentially dense in $\mathrm{USC}_p^\star(X)$.*

Proof: One can easily see that $f_{U,g} + g \geq g$ and $f_{U,g} + g$ is bounded upper semicontinuous for $g \in \mathrm{USC}_p^\star(X)$.

We show that if $\mathcal{U}$ is an open ω-cover then $S(\mathcal{U})$ is upper dense in $\mathrm{USC}_p^\star(X)$. Assume that $g \in \mathrm{USC}_p^\star(X)$. If V is the neighborhood of g defined by (3), then there exists a $U \in \mathcal{U}$ such that $x_0, \ldots, x_k \in U$. Then $f_{U,g}(x_j) + g(x_j) = g(x_j)$ for $j = 0, \ldots, k$. Hence $f_{U,g} + g \in V$, $f_{U,g} + g \geq g$ and $(f_{U,g} + g) - g \in \mathrm{USC}_p^\star(X)$. Thus $S(\mathcal{U})$ is upper dense.

If $\mathcal{U}$ is an open γ-cover then $S(\mathcal{U})$ is upper sequentially dense in $\mathrm{USC}_p^\star(X)$. Indeed, let $g \in \mathrm{USC}_p^\star(X)$. Let $\{U_i : i \in \omega\}$ be a countable γ-subcover of $\mathcal{U}$. For $i \in \omega$, we let $g_i = f_{U_i,g} + g \in S(\mathcal{U})$. We show that the sequence $\langle g_i : i \in \omega \rangle$ converges to g. Let V be a neighborhood of g defined by (3). Since $\{U_i : i \in \omega\}$ is a γ-cover, there exists an i_0 such that $x_0, \ldots, x_k \in U_i$ for $i \geq i_0$. Then for such an i we have $g_i(x_j) = g(x_j)$ for $j = 0, \ldots, k$. Therefore the elements of the sequence $\langle g_i : i \in \omega \rangle$ belong to V for $i \geq i_0$. As above, $g_i - g \in \mathrm{USC}_p^\star(X)$. Thus $S(\mathcal{U})$ is upper sequentially dense. $\square$

Theorem 5.2. *Let $\Phi = \Omega, \Gamma$. Then the following are equivalent:*

a) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \mathcal{D})$.

b) $\mathrm{USC}_p^\star(X)$ *is separable and the topological space X possesses the covering property* $\mathrm{S}_1(\Phi, \Omega)$.

c) $\mathrm{USC}_p^\star(X)$ *is separable and satisfies the selection principle* $\mathrm{S}_1(\Phi_h, \Omega_h)$ *for every* $h \in \mathrm{USC}_p^\star(X)$.

d) $\mathrm{USC}_p^\star(X)$ *is separable and satisfies the selection principle* $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \Omega_h)$ *for every* $h \in \mathrm{USC}_p^\star(X)$.

Proof:
a) $\to$ b). Let $\{\mathcal{U}_n : n \in \omega\} \subseteq \Phi$. We may assume that $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ for each $n \in \omega$. If $\Phi = \Gamma$ we may also assume that for every $n \in \omega$, the cover $\mathcal{U}_n$ is a countable family $\{U_i^n : i \in \omega\}$.

For every $n \in \omega$ we set

$$S_n = S(\mathcal{U}_n). \tag{8}$$

By Lemma 5.1 we have $S_n \in \tilde{\Phi}^\uparrow$. Thus, by the selection principle $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \mathcal{D})$, for every $n \in \omega$ we obtain an $f_{U_n,h_n} \in S_n$ such that $\{f_{U_n,h_n} : n \in \omega\}$ is dense in $\mathrm{USC}_p^\star(X)$. We show that $\{U_n : n \in \omega\}$ is an ω-cover.

Let $x_0, \ldots, x_k \in X$. Consider the open non-empty set

$$U = \{g \in \mathrm{USC}_p^\star(X) : |g(x_j)| < 1/2 \text{ for } j = 0, \ldots, k\}$$

Since the set $\{f_{U_n,h_n} : n \in \omega\}$ is dense in $\mathrm{USC}_p^\star(X)$, there exists an n such that $f_{U_n,h_n} \in U$. Since $|f_{U_n,h_n}(x_i)| < 1/2$ for $i = 0, \ldots, k$, by (7) we obtain $x_0, \ldots, x_k \in U_n$.

The implication b) $\to$ c) follows by Corollary 3.2.

The implication c) $\to$ d) is obvious by (4).

d) $\to$ a). We assume that $\mathrm{USC}_p^\star(X)$ is separable and satisfies the selection principle $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \Omega_h)$ for every $h \in \mathrm{USC}_p^\star(X)$. Thus, there exists a countable set $D = \{d_n : n \in \omega\}$ dense in $\mathrm{USC}_p^\star(X)$. Let $\{S_{n,m} : n, m \in \omega\}$ be a sequence of subsets of $\mathrm{USC}_p^\star(X)$ such that $S_{n,m} \in \tilde{\Phi}^\uparrow$ for each $n, m \in \omega$. For every $n \in \omega$ we apply the sequence selection principle $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \Omega_{d_n})$ to the sequence $\langle S_{n,m} : m \in \omega\rangle$ and for every $m \in \omega$ we obtain $d_{n,m} \in S_{n,m}$ such that $d_n \in \overline{\{d_{n,m} : m \in \omega\}}$. Then $\{d_{n,m} : n, m \in \omega\}$ is dense in $\mathrm{USC}_p^\star(X)$. $\square$

By Theorem 3.1 and analogously to the proof of Theorem 5.2 we get the following theorem.

Theorem 5.3. *Let $\Phi = \Omega$ or $\Phi = \Gamma$. Assume that $\mathrm{C}_p^\star(X)$ is countably dense in $\mathrm{USC}_p^\star(X)$. Then for any couple $\langle\Phi, \Psi\rangle$ different from $\langle\Omega, \mathcal{O}\rangle$, the following are equivalent:*

a) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \mathcal{D})$*,*

b) $\mathrm{USC}_p^\star(X)$ *is separable and the topological space X possesses the covering property* $\mathrm{S}_1(\Phi, \Omega)$*.*

c) $\mathrm{USC}_p^\star(X)$ *is separable and satisfies the selection principle* $\mathrm{S}_1(\Phi_{\mathbf{0}}, \Omega_{\mathbf{0}})$*.*

d) $\mathrm{USC}_p^\star(X)$ *is separable and satisfies the selection principle* $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \Omega_{\mathbf{0}})$*.*

Proof:

We prove only the implication d) $\to$ a). The proofs of other implications are almost equal to those in the proof of Theorem 5.2.

Assume that $\mathrm{C}_p^\star(X)$ is countably dense in $\mathrm{USC}_p^\star(X)$, $\mathrm{USC}_p^\star(X)$ is separable and satisfies the selection principle $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \Omega_{\mathbf{0}})$. Thus, there exists a countable set $D = \{d_n : n \in \omega\}$ dense in $\mathrm{USC}_p^\star(X)$. Since $\mathrm{C}_p^\star(X)$ is countably dense in $\mathrm{USC}_p^\star(X)$, for every $n \in \omega$ there exists a countable set $D_n = \{d_{n,m} : m \in \omega\} \subseteq \mathrm{C}_p^\star(X)$ such that $d_n \in \overline{D_n}$ for each $n \in \omega$.

Let $\{S_{n,m,l} : n, m, l \in \omega\}$ be a sequence of subsets of $\mathrm{USC}_p^\star(X)$, each $S_{n,m,l}$ being in $\tilde{\Phi}^\uparrow$. We can apply the sequence selection principle $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \Omega_{\mathbf{0}})$ to the sequence

$$\langle \{h - d_{n,m} : h \in S_{n,m,l}\} : l \in \omega\rangle \,.$$

For every $l \in \omega$ we obtain $d_{n,m,l} \in S_{n,m,l}$ such that

$$\mathbf{0} \in \overline{\{d_{n,m,l} - d_{n,m} : l \in \omega\}}.$$

Then

$$d_{n,m} \in \overline{\{d_{n,m,l} : l \in \omega\}}.$$

Thus $\{d_{n,m,l} : n, m, l \in \omega\}$ is the desired countable dense set. $\square$

Since no (infinite Hausdorff) topological space has the covering property $\mathrm{S}_1(\mathcal{O}, \Omega)$, we obtain

Theorem 5.4. $\mathrm{USC}_p^\star(X)$ *does not have the property* $\mathrm{S}_1(\mathcal{P},\mathcal{D})$ *for any topological space* X.

Proof: As in proof of Theorem 5.2 one can show that if $\mathrm{USC}_p^\star(X)$ possesses the property $\mathrm{S}_1(\mathcal{P},\mathcal{D})$ then X possesses the covering property $\mathrm{S}_1(\mathcal{O},\Omega)$. $\square$

6. Sequentially dense selectors of $\mathbf{USC}_{\mathbf{p}}^\star\mathbf{(X)}$

Theorem 6.1. *Let* $\Phi=\Omega$ *or* $\Phi=\Gamma$. *Then for any couple* $\langle\Phi,\Psi\rangle$ *different from* $\langle\Omega,\mathcal{O}\rangle$, *the following are equivalent:*

a) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Phi}^\uparrow,\mathcal{S})$.

b) $\mathrm{USC}_p^\star(X)$ *is sequentially separable and the topological space* X *possesses the covering property* $\mathrm{S}_1(\Phi,\Gamma)$.

c) $\mathrm{USC}_p^\star(X)$ *is sequentially separable and satisfies the selection principle* $\mathrm{S}_1(\Phi_h,\Gamma_h)$ *for every* $h\in\mathrm{USC}_p^\star(X)$.

d) $\mathrm{USC}_p^\star(X)$ *is sequentially separable and satisfies the selection principles* $\mathrm{S}_1(\Gamma_{\mathbf{0}},\Gamma_{\mathbf{0}})$ *and* $\mathrm{S}_1(\tilde{\Phi}^\uparrow,\Gamma_h)$ *for every* $h\in\mathrm{USC}_p^\star(X)$.

Proof:
a) $\to$ b). Let $\{\mathcal{U}_n:n\in\omega\}\subseteq\Phi$. We may assume that $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ for each $n\in\omega$. If $\Phi=\Gamma$ we may also assume that for every $n\in\omega$, the cover $\mathcal{U}_n=\{U_i^n:i\in\omega\}$ is a countable family.

We define the sets S_n by (8). By Lemma 5.1, $S_n\in\tilde{\Phi}^\uparrow(\mathrm{USC}_p^\star(X))$. We apply the selection principle $\mathrm{S}_1(\tilde{\Phi}^\uparrow,\mathcal{S})$ and for every n we obtain a function $f_{U_n,h_n}\in S_n$ such that $\{f_{U_n,h_n}:n\in\omega\}$ is sequentially dense in $\mathrm{USC}_p^\star(X)$. For every n we shall find a set $V_n\in\mathcal{U}_n$ such that $\{V_n:n\in\omega\}$ is a γ-cover.

Evidently there exists an increasing sequence $\langle n_k:k\in\omega\rangle$ such that $f_{U_{n_k},h_{n_k}}\to\mathbf{0}$. We set $V_{n_k}=U_{n_k}$. If $n_k<n<n_{k+1}$, then by (1) we can find a set $V_n\in\mathcal{U}_n$ such that $U_{n_{k+1}}\subseteq V_n$.

Let $x\in X$. Since

$$W=\{g\in\mathrm{USC}_p^\star(X):|g(x)|<1\}$$

is a neighborhood of $\mathbf{0}$, there exists an k_0 such that $f_{U_{n_k},h_{n_k}}\in W$ for each $k\geq k_0$. If $g\in W$ then $g(x)<1$. Thus for $k\geq k_0$ we have $f_{U_{n_k},h_{n_k}}(x)<1$. Therefore $x\in U_{n_k}=V_{n_k}$. By the choose of V_n for $n\notin\{n_k:k\in\omega\}$ we obtain that $x\in V_n$ for each $n\geq n_{k_0}$.

The implication b) $\to$ c) follows by Corollary 3.2.

The implication c) $\to$ d) is obvious by (4).

We prove the implication d) $\to$ a).

Assume that there exists a countable set $\{d_n:n\in\omega\}\subseteq\mathrm{USC}_p^\star(X)$ sequentially dense in $\mathrm{USC}_p^\star(X)$ and $\mathrm{USC}_p^\star(X)$ satisfies the selection principles $\mathrm{S}_1(\Gamma_{\mathbf{0}},\Gamma_{\mathbf{0}})$ and $\mathrm{S}_1(\tilde{\Phi}^\uparrow,\Gamma_h)$ for each $h\in\mathrm{USC}_p^\star(X)$.

Let $(S_{n,m} : n, m \in \omega)$ be a sequence of subsets of $\mathrm{USC}_p^\star(X)$ all being in $\tilde{\Phi}^\uparrow$. For every n we apply the selection principle $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \Gamma_{d_n})$ to the sequence $(S_{n,m} : m \in \omega)$. Then for every $m \in \omega$ we obtain $d_{n,m} \in S_{n,m}$, $d_{n,m} \geq d_n$, $d_{n,m} - d_n \in \mathrm{USC}_p(X)$ and such that $d_{n,m} - d_n \to \mathbf{0}$ $(m \to \infty)$.

We show that the set $\{d_{n,m} : n, m \in \omega\}$ is the desired sequentially dense set.

Indeed, if $h \in \mathrm{USC}_p^\star(X)$ then there exists an increasing sequence $\langle n_k : k \in \omega\rangle$ such that $d_{n_k} \to h$. Since for every k we have

$$d_{n_k,m} - d_{n_k} \to \mathbf{0},$$

by $\mathrm{S}_1(\Gamma_\mathbf{0}, \Gamma_\mathbf{0})$ there exists a sequence $\langle m_k : k \in \omega\rangle$ such that

$$d_{n_k,m_k} - d_{n_k} \to \mathbf{0}.$$

Thus

$$d_{n_k,m_k} = (d_{n_k,m_k} - d_{n_k}) + d_{n_k}$$

converges to h $(k \to \infty)$. □

Corollary 6.2. *Let $\Phi = \Omega$ or $\Phi = \Gamma$. Assume that X is a Tychonoff topological space and $\mathrm{C}_p^\star(X)$ is seuqentially dense on $\mathrm{USC}_p^\star(X)$. Then the following are equivalent:*

a) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \mathcal{S})$.

b) *The topological space X possesses the OB-property and the covering property* $\mathrm{S}_1(\Phi, \Gamma)$.

c) $\mathrm{USC}_p^\star(X)$ *is sequentially separable and satisfies the selection principle* $\mathrm{S}_1(\Phi_h, \Gamma_h)$ *for each* $h \in \mathrm{USC}_p^\star(X)$.

d) $\mathrm{USC}_p^\star(X)$ *is sequentially separable and satisfies the selection principles* $\mathrm{S}_1(\Gamma_\mathbf{0}, \Gamma_\mathbf{0})$ *and* $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \Gamma_h)$ *for each* $h \in \mathrm{USC}_p^\star(X)$.

Proof: The only non-trivial implication is "b) of Theorem 6.1" to "b) of the Corollary". By $\mathrm{S}_1(\Phi, \Gamma)$ we obtain that $\mathrm{USC}_p^\star(X)$ satisfies the selection principle $\mathrm{S}_1(\Gamma_\mathbf{0}, \Gamma_\mathbf{0})$. Thus we obtain a countable subset of $\mathrm{C}_p^\star(X)$ dense in $\mathrm{USC}_p^\star(X)$. Since X is Tychonoff we can apply Theorem 4.3. □

Corollary 6.3. *Let $\Phi = \Omega$ or $\Phi = \Gamma$. Assume that the Tychonoff topological space X has the OB-property. Then for the following are equivalent:*

a) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \mathcal{S})$.

b) *X possesses the covering property* $\mathrm{S}_1(\Phi, \Gamma)$.

c) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\Phi_h, \Gamma_h)$ *for every* $h \in \mathrm{USC}_p^\star(X)$.

d) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principles* $\mathrm{S}_1(\Gamma_\mathbf{0}, \Gamma_\mathbf{0})$ *and* $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \Gamma_\mathbf{0})$.

Proof: We have only to prove the implication d) $\to$ a). By Theorem 4.3 there exists a countable set $\{d_n : n \in \omega\} \subseteq \mathrm{C}_p^\star(X)$ sequentially dense in $\mathrm{USC}_p^\star(X)$.

Let $(S_{n,m} : n, m \in \omega)$ be a sequence of subsets of $\mathrm{USC}_p^\star(X)$ all in $\tilde{\Phi}^\uparrow$. For every n we apply the selection principle $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \Gamma_{\mathbf{0}})$ to the sequence $(S_{n,m} - d_n : m \in \omega)$. Then for every $m \in \omega$ we obtain $d_{n,m} \in S_{n,m}$, $d_{n,m} \geq d_n$, $d_{n,m} - d_n \in \mathrm{USC}_p(X)$ and such that $d_{n,m} - d_n \to \mathbf{0}$ $(m \to \infty)$.

We can show that the set $\{d_{n,m} : n, m \in \omega\}$ is sequentially dense subset of $\mathrm{USC}_p^\star(X)$ equally as in the proof of Theorem 6.1. $\square$

Since no infinite Hausdorff topological space satisfies $\mathrm{S}_1(\mathcal{O}, \Gamma)$ as above one can easily prove

Theorem 6.4. $\mathrm{USC}_p^\star(X)$ *does not have the property* $\mathrm{S}_1(\mathcal{P}, \mathcal{S})$ *for any (infinite Hausdorff) topological space* X.

7. Pointwise dense selectors of $\mathbf{USC_p^*(X)}$

Theorem 7.1. *For* $\Phi = \Gamma$ *the following are equivalent:*

a) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \mathcal{P})$.

b) *The topological space* X *possesses the covering property* $\mathrm{S}_1(\Phi, \mathcal{O})$.

c) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\Phi_{\mathbf{0}}, \mathcal{O}_{\mathbf{0}})$.

d) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \mathcal{O}_{\mathbf{0}})$.

If X *possesses the property* (ε) *then the equivalences hold true also for* $\Phi = \Omega$.

Proof:

a) $\to$ b). Let $\{\mathcal{U}_n : n \in \omega\} \in \Phi$. We can assume that for every $n \in \omega$ the cover $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$. If $\Phi = \Gamma$ we may also assume that for every $n \in \omega$, the cover $\mathcal{U}_n$ is a countable family $\{U_i^n : i \in \omega\}$.

We define the sets S_n by (8). By Lemma 5.1, every S_n is upper sequentially dense and pointwise dense in $\mathbb{R}$. By the selection principle $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \mathcal{P})$, for every $n \in \omega$ we obtain an $f^n_{U_n,h_n} \in S_n$ such that the set $\{f^n_{U_n,h_n} : n \in \omega\}$ is pointwise dense in $\mathbb{R}$. We show that $\{U_n : n \in \omega\}$ is a cover.

If $x \in X$ then there exists an n such that $f^n_{U_n,h_n}(x) \in (-1/2, 1/2)$. Then $x \in U_n$.

By Corollary 3.2 we obtain b) $\to$ c). The implication c) $\to$ d) is trivial. We show the implication d) $\to$ a).

Let $\{S_{n,m} : n, m \in \omega\}$ be a sequence of upper sequentially dense subsets of $\mathrm{USC}_p^\star(X)$. Let the rational numbers $\mathbb{Q}$ be enumerated as $\{q_n : n \in \omega\}$. Fix $n \in \omega$. By $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \mathcal{O}_{\mathbf{0}})$ for every $m \in \omega$ there exists $f_{n,m} \in S_{n,m}$, $f_{n,m} \geq q_n$ such that $\{f_{n,m} - q_n : m \in \omega\} \in \mathcal{O}_{\mathbf{0}}$. Then $\{f_{n,m} : n, m \in \omega\}$ is pointwise dense in $\mathbb{R}$. $\square$

Note 7.2. *By Theorem 3.3 and Corollary 3.4 similar results hold true also for the principle* S_{fin}.

8. Dense selectors of the space $\mathrm{C}_p(X)$

In papers [1, 12, 14, 15, 17, 22] the authors have got characterizations of dense, sequentially dense and pointwise dense selectors of the space of real-valued continuous functions $C_p(X)$ for Tychonoff space X. We can show similar results for selectors of sequences of dense, sequentially dense and pointwise dense subsets of $\mathrm{C}_p^\star(X)$ for a normal topological space X.

In the next we assume that X is a normal topological space.

Let $\mathcal{U}$ be a shrinkable cover of X, $\mathcal{W}$ being a cover of X such that (2) holds true. We set

$$T(\mathcal{U}) = \{g_{W,h} + h : W \in \mathcal{W} \wedge h \in \mathrm{C}_p^\star(X)\}, \tag{9}$$

where $f_{W,h} \in \mathrm{C}_p^\star(X)$ is such that

$$g_{W,h}(x) = \begin{cases} 0 & \text{if } x \in W, \\ 1 + \sup|h| & \text{if } x \in X \setminus U_W. \end{cases} \tag{10}$$

The existence of such a function follows from that that X is a normal topological space.

Equally as in Section 5 one can easily prove

Lemma 8.1.

1) *If $\mathcal{U}$ is an open shrinkable ω-cover, then the family $S(\mathcal{U})$ is upper dense in $\mathrm{C}_p^\star(X)$.*

2) *If $\mathcal{U}$ is an open shrinkable γ-cover, then the family $S(\mathcal{U})$ is upper sequentially dense in $\mathrm{C}_p^\star(X)$.*

Theorem 8.2. *Let $\Phi = \Omega$ or $\Phi = \Gamma$. Then for a normal topological space X the following are equivalent:*

a) *$\mathrm{C}_p^\star(X)$ satisfies the selection principle $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \mathcal{D})$.*

b) *$iw(X) = \aleph_0$ and the topological space X possesses the covering property $\mathrm{S}_1(\Phi^{sh}, \Omega)$.*

c) *$\mathrm{C}_p^\star(X)$ is separable and satisfies the selection principle $\mathrm{S}_1(\Phi_{\mathbf{0}}, \Omega_{\mathbf{0}})$.*

d) *$\mathrm{C}_p^\star(X)$ is separable and satisfies the selection principle $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \Omega_{\mathbf{0}})$.*

Proof: We show a) $\to$ b). Let $\langle \mathcal{U}_n : n \in \omega\rangle$ be a sequence of open shrinkable φ-covers of X. Let $\langle \mathcal{W}_n : n \in \omega\rangle$ be a sequence of open φ-covers such that (2) holds true for each n. We can assume that $\mathcal{U}_n = \{U_W : W \in \mathcal{W}_n\}$, where U_W is the set of (2). For every $n \in \omega$ we set

$$T_n = T(\mathcal{U}_n). \tag{11}$$

By Lemma 8.1 we can apply $\mathrm{S}_1(\tilde{\Phi}^{\uparrow}, \mathcal{D})$ to the sequence $\langle T_n : n \in \omega \rangle$ and we obtain for every $n \in \omega$ a function $g_n \in T_n$ such that the set $\{g_n : n \in \omega\}$ is dense in $\mathrm{C}_p(X)$. By definition of T_n, there exists $h_n \in \mathrm{C}_p(X)$ and $W_n \in \mathcal{W}_n$ such that $g_n = g_{W_n,h_n} + h_n$. We show that $\{U_{W_n} : n \in \omega\}$ is an ω-cover.

Let $x_0, \dots, x_k \in X$. Let us consider the neighborhood of $\mathbf{0}$ defined as

$$V = \{h \in \mathrm{C}_p(X) : |h(x_i)| < \frac{1}{2} \text{ for } i = 0, \dots, k\}.$$

Since $\{g_n : n \in \omega\}$ is dense in $\mathrm{C}_p(X)$ there exists an $n \in \omega$ such that $g_n \in V$. We know that $g_n = g_{W_n,h_n} + h_n$. For $i = 0, \dots, k$ we have $|g_n(x_i)| < \frac{1}{2}$. If $x \in X \setminus U_{W_n}$ then $g_n(x) \geq 1$. Thus $x_0, \dots, x_k \in U_{W_n}$.

Similarly to the proof of Theorem 5.2 the implication b) $\to$ c) is special case of Theorem 3.3 and the implication c) $\to$ d) is trivial. We show d) $\to$ a).

Assume that $\mathrm{C}_p(X)$ is separable and satisfies the selection principle $\mathrm{S}_1(\tilde{\Phi}^{\uparrow}, \Omega_{\mathbf{0}})$. Let $\{d_n : n \in \omega\}$ be a dense subset of $\mathrm{C}_p(X)$. Let $\{S_{n,m} : n, m \in \omega\}$ be a sequence of subsets of $\mathrm{C}_p(X)$, each $S_{n,m}$ being in $\tilde{\Phi}^{\uparrow}$. We can apply the sequence selection principle $\mathrm{S}_1(\tilde{\Phi}^{\uparrow}, \Omega_{\mathbf{0}})$ to the sequence

$$\langle \{h - d_n : h \in S_{n,m}\} : m \in \omega \rangle \,.$$

For every $m \in \omega$ we obtain $d_{n,m} \in S_{n,m}$ such that

$$\mathbf{0} \in \overline{\{d_{n,m} - d_n : m \in \omega\}}.$$

Then

$$d_n \in \overline{\{d_{n,m} : m \in \omega\}}.$$

Thus the selector $\{d_{n,m} : n, m \in \omega\}$ is a countable dense subset of $\mathrm{C}_p^{\star}(X)$, every $d_{n,m}$ belongs to $S_{n,m}$. □

Modifying the proof of Theorem 6.1 similarly as the proof of Theorem 8.2, one can easily prove the next two results. Namely

Theorem 8.3. *For $\Phi = \mathcal{O}$, Ω, Γ and a normal topological space X the following are equivalent:*

a) $\mathrm{C}_p^{\star}(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Phi}^{\uparrow}, \mathcal{S})$.

b) *The topological space X possesses the V-property and the covering property* $\mathrm{S}_1(\Phi^{sh}, \Gamma)$.

c) $\mathrm{C}_p^{\star}(X)$ *is sequentially separable and satisfies the selection principle* $\mathrm{S}_1(\Phi_{\mathbf{0}}, \Gamma_{\mathbf{0}})$.

d) $\mathrm{C}_p^{\star}(X)$ *is sequentially separable and satisfies the selection principle* $\mathrm{S}_1(\tilde{\Phi}^{\uparrow}, \Gamma_{\mathbf{0}})$.

and

Theorem 8.4. *For $\Phi = \mathcal{O}$, Γ and a normal topological space X the following are equivalent:*

a) $\mathrm{C}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \mathcal{P})$.

b) X *possesses the covering property* $\mathrm{S}_1(\Phi^{sh}, \mathcal{O})$.

c) $\mathrm{C}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\Phi_{\mathbf{0}}, \mathcal{O}_{\mathbf{0}})$.

d) $\mathrm{C}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Phi}^\uparrow, \mathcal{O}_{\mathbf{0}})$.

If X possesses the property (ε) then the equivalences hold true also for $\Phi = \Omega$.

Note 8.5. *Similar results hold true also for the principle S_{fin}.*

9. Some consequences

Since in a regular topological space every open ω-cover is shrinkable, see e.g. [3], Lemma 7.1, in b) of Theorems 8.2, 8.3 and 8.4 we can replace the corresponding covering principle $\mathrm{S}_1(\Omega^{sh}, \Psi)$ by $\mathrm{S}_1(\Omega, \Psi)$. Hence we obtain, e.g. from Theorems 5.2 and 8.2, the following

Corollary 9.1. *For $\Psi = \Omega, \Gamma$ and a normal topological space X the following statements are equivalent:*

a) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Omega}^\uparrow, \tilde{\Psi})$.

b) *The topological space X possesses the OB-property and the covering property* $\mathrm{S}_1(\Omega, \Psi)$.

c) $\mathrm{USC}_p^\star(X)$ *is separable and satisfies the selection principle* $\mathrm{S}_1(\Omega_{\mathbf{0}}, \Psi_{\mathbf{0}})$.

d) $\mathrm{USC}_p^\star(X)$ *is separable and satisfies the selection principle* $\mathrm{S}_1(\tilde{\Omega}^\uparrow, \Psi_{\mathbf{0}})$.

e) $\mathrm{C}_p(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Omega}^\uparrow, \tilde{\Psi})$.

f) $\mathrm{C}_p(X)$ *is separable and satisfies the selection principle* $\mathrm{S}_1(\Omega_{\mathbf{0}}, \Psi_{\mathbf{0}})$.

g) $\mathrm{C}_p(X)$ *is separable and satisfies the selection principle* $\mathrm{S}_1(\tilde{\Omega}^\uparrow, \Psi_{\mathbf{0}})$.

Since $\mathrm{USC}_p^\star(X)$ and $\mathrm{C}_p^\star(X)$ are (sequentially) separable for a separable metrizable space X (see Theorems 4.1 and 4.3), we get the following result.

Corollary 9.2. *For $\Psi = \Omega, \Gamma$ and a separable metrizable space X the following statements are equivalent:*

a) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Omega}^\uparrow, \tilde{\Psi})$.

b) X *possesses the covering property* $\mathrm{S}_1(\Omega, \Psi)$.

c) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\Omega_{\mathbf{0}}, \Psi_{\mathbf{0}})$.

d) $\mathrm{USC}_p^\star(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Omega}^\uparrow, \Psi_{\mathbf{0}})$.

e) $\mathrm{C}_p(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Omega}^\uparrow, \tilde{\Psi})$.

f) $\mathrm{C}_p(X)$ *satisfies the selection principle* $\mathrm{S}_1(\Omega_{\mathbf{0}}, \Psi_{\mathbf{0}})$.

g) $\mathrm{C}_p(X)$ *satisfies the selection principle* $\mathrm{S}_1(\tilde{\Omega}^\uparrow, \Psi_{\mathbf{0}})$.

Let us recall that a set $F \subseteq X$ is a **zero set** if there exists a continuous function $f \in \mathrm{C}_p(X)$ such that $F = \{x \in X : f(x) = 0\}$. A complement of a zero set is a **cozero set**. We denote by $\Phi_{cz}(X)$ the family of all φ-covers of X consisting of cozero sets. A φ-cover $\mathcal{U}$ is **functionally shrinkable** if there exists a φ-cover $\mathcal{W}$ consisting of*/ zero sets such that (2) holds true. The family of all functionally shrinkable φ-covers of X will be denoted by $\Phi^{fsh}(X)$. We shall be especially interested in the families $\Phi_{cz}^{fsh}(X)$.

One can easily see that Theorem 3.5 may be modified as

Theorem 9.3. *Assume that* Φ *is one of the symbols* Ω, Γ *and* Ψ *is one of the symbols* $\mathcal{O}$, Ω, Γ. *Then for any couple* $\langle\Phi, \Psi\rangle$ *different from* $\langle\Omega, \mathcal{O}\rangle$, *a topological space* X *is an* $\mathrm{S}_1(\Phi_{cz}^{fsh}, \Psi)$*-space if and only if* $\mathrm{C}_p^\star(X)$ *satisfies the selection principle* $S_1(\Phi_{\mathbf{0}}, \Psi_{\mathbf{0}})$.

If a topological space X *possesses the property* (ε) *then the equivalence holds true also for the couple* $\langle\Omega, \mathcal{O}\rangle$.

As a consequence, if we replace in parts b) of Theorems 8.2, 8.3 and 8.4 the covering property $\mathrm{S}_1(\Phi^{sh}, \Psi)$ by the covering property $\mathrm{S}_1(\Phi_{cz}^{fsh}, \Psi)$ then we can omit the condition that the topological space X is normal.

10. Remarks

The idea to use upper semicontinuous functions for expressing some covering properties of a topological space arose in [2]. That was M. Sakai [19] who immediately exploited this idea. The starting point of our study presented in this paper were the results of [3] and the investigation started in [12].

Some equivalences of our Theorems are already known. M. Scheepers [22] has prove that a metric separable space X is an $\mathrm{S}_1(\Omega, \Omega)$-space if and only if $\mathrm{C}_p(X)$ satisfies $\mathrm{S}_1(\mathcal{D}, \mathcal{D})$. A.V. Osipov [12] proves several characterizations of topological spaces X with $\mathrm{C}_p(X)$ satisfying $\mathrm{S}_1(\mathcal{D}, \mathcal{S})$, $\mathrm{S}_1(\mathcal{S}, \mathcal{D})$ or $\mathrm{S}_1(\mathcal{S}, \mathcal{S})$.

In [3] similar results are presented for S_{fin}. Consequently, one can easily see that plenty of our results may be proved also for S_{fin}.

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